

RESOLUÇÃO DE INTEGRAIS POR SUBSTITUIÇÃO

a) $\int \sec^2 5x dx$

Resolução

Fazemos $5x = u \Rightarrow (5x)' = \frac{du}{dx} \Rightarrow 5 = \frac{du}{dx} \Rightarrow 5dx = du \Rightarrow dx = \frac{1}{5} du$

Logo:

$$\int \sec^2 5x dx = \int \sec^2 u \cdot \frac{1}{5} du = \frac{1}{5} \int \sec^2 u du = \frac{1}{5} \operatorname{tg} u + c$$

Voltando u para x , temos: $\int \sec^2 5x dx = \frac{1}{5} \operatorname{tg} (5x) + c$

b) $\int \operatorname{ctg} x dx$

Resolução

Observe que $\operatorname{ctg} x = \frac{\cos x}{\operatorname{sen} x}$, assim temos que:

$$\int \operatorname{ctg} x dx = \int \frac{\cos x}{\operatorname{sen} x} dx = \int \frac{1}{\operatorname{sen} x} \cos x dx$$

Agora fazemos $\operatorname{sen} x = u \Rightarrow (\operatorname{sen} x)' = \frac{du}{dx} \Rightarrow \cos x = \frac{du}{dx} \Rightarrow \cos x dx = du$

E finalmente:

$$\int \operatorname{ctg} x dx = \int \frac{\cos x}{\operatorname{sen} x} dx = \int \frac{1}{\operatorname{sen} x} \cos x dx = \int \frac{1}{u} du = \ln|u| + c = \ln|\operatorname{sen} x| + c$$

c) $\int \frac{x^3 - 6x + 5}{x} dx$

Resolução

$$\int \frac{x^3 - 6x + 5}{x} dx = \int \left(\frac{x^3}{x} - \frac{6x}{x} + \frac{5}{x} \right) dx = \int \left(x^2 - 6 + \frac{5}{x} \right) dx =$$

$$\int x^2 dx - 6 \int dx + 5 \int \frac{1}{x} dx = \frac{x^3}{3} - 6x + 5 \ln|x| + c$$

d) $\int \sqrt{ax-b} dx$

Resolução

Fazendo $ax-b = u \Rightarrow (ax-b)' = \frac{du}{dx} \Rightarrow a = \frac{du}{dx} \Rightarrow adx = du \Rightarrow dx = \frac{1}{a} du$

$$\int \sqrt{ax-b} dx = \int \sqrt{u} \frac{1}{a} du = \frac{1}{a} \int u^{\frac{1}{2}} du = \frac{1}{a} \cdot \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{a} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c =$$

$$= \frac{1}{a} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + c = \frac{2}{3a} \sqrt{u^3} + c = \frac{2}{3a} \sqrt{(ax-b)^3} + c$$

e) $\int \operatorname{sen} x \cdot \cos x dx$

Resolução

Fazemos $\operatorname{sen} x = u \Rightarrow (\operatorname{sen} x)' = \frac{du}{dx} \Rightarrow \cos x = \frac{du}{dx} \Rightarrow \cos x dx = du$

$$\int \operatorname{sen} x \cdot \cos x dx = \int u du = \frac{u^{1+1}}{1+1} + c = \frac{u^2}{2} + c = \frac{(\operatorname{sen} x)^2}{2} + c = \frac{\operatorname{sen}^2 x}{2} + c$$

f) $\int \frac{x}{1+x^2} dx$

Resolução

Observe que $\int \frac{x}{1+x^2} dx = \int \frac{1}{1+x^2} x dx$

Fazendo $1+x^2 = u \Rightarrow (1+x^2)' = \frac{du}{dx} \Rightarrow 2x = \frac{du}{dx} \Rightarrow 2x dx = du \Rightarrow$

$xdx = \frac{1}{2} du$

Logo:

$$\int \frac{x}{1+x^2} dx = \int \frac{1}{1+x^2} x dx = \int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln|1+x^2| + c$$

g) $\int \frac{x}{1+x^4} dx$

Resolução

Observe que $\int \frac{x}{1+x^4} dx = \int \frac{1}{1+x^4} x dx = \int \frac{1}{1+(x^2)^2} x dx$

Fazendo $x^2 = u \Rightarrow (x^2)' = \frac{du}{dx} \Rightarrow 2x = \frac{du}{dx} \Rightarrow 2x dx = du \Rightarrow x dx = \frac{1}{2} du$

$$\text{Daí } \int \frac{x}{1+x^4} dx = \int \frac{1}{1+x^4} x dx = \int \frac{1}{1+(x^2)^2} x dx = \int \frac{1}{1+u^2} \cdot \frac{1}{2} du = \frac{1}{2} \int \frac{1}{1+u^2} du =$$

$$= \frac{1}{2} \operatorname{arc} \operatorname{tg} u + c = \frac{1}{2} \operatorname{arc} \operatorname{tg} (x^2) + c$$

h) $\int \frac{e^t}{1+e^{2t}} dt$

Resolução

Observe que $\int \frac{e^t}{1+e^{2t}} dt = \int \frac{1}{1+(e^t)^2} e^t dt$

Daí, fazemos: $e^t = u \Rightarrow (e^t)' = \frac{du}{dt} \Rightarrow e^t = \frac{du}{dt} \Rightarrow e^t dt = du$

Assim: $\int \frac{e^t}{1+e^{2t}} dt = \int \frac{1}{1+(e^t)^2} e^t dt = \int \frac{1}{1+u^2} du = \text{arc tg } u + c = \text{arc tg } (e^t) + c$

i) $\int \frac{\sec x \cdot \text{tg } x}{\sqrt{1-\sec^2 x}} dx$

Resolução

Observe que

$$\int \frac{\sec x \cdot \text{tg } x}{\sqrt{1-\sec^2 x}} dx = \int \frac{1}{\sqrt{1-\sec^2 x}} \sec x \cdot \text{tg } x dx = \int \frac{1}{\sqrt{1-(\sec x)^2}} \sec x \cdot \text{tg } x dx$$

Fazendo $\sec x = u \Rightarrow (\sec x)' = \frac{du}{dx} \Rightarrow \sec x \cdot \text{tg } x = \frac{du}{dx} \Rightarrow$

$$\sec x \cdot \text{tg } x dx = du$$

Assim, temos que:

$$\int \frac{\sec x \cdot \text{tg } x}{\sqrt{1-\sec^2 x}} dx = \int \frac{1}{\sqrt{1-\sec^2 x}} \sec x \cdot \text{tg } x dx = \int \frac{1}{\sqrt{1-(\sec x)^2}} \sec x \cdot \text{tg } x dx =$$

$$\int \frac{1}{\sqrt{1-u^2}} du = \text{arc sen } u + c = \text{arc sen } (\sec x) + c$$

j) $\int a^{5x} dx$

Resolução

Fazendo $5x = u \Rightarrow (5x)' = \frac{du}{dx} \Rightarrow 5 = \frac{du}{dx} \Rightarrow 5dx = du \Rightarrow dx = \frac{1}{5} du$

$$\int a^{5x} dx = \int a^u \cdot \frac{1}{5} du = \frac{1}{5} \int a^u du = \frac{1}{5} \cdot \frac{a^u}{\ln a} + c = \frac{1}{5} \cdot \frac{a^{5x}}{\ln(5x)} + c$$

k) $\int a^x \cdot e^x dx$

Resolução

Observe que

$$\int a^x \cdot e^x dx = \int (a \cdot e)^x dx$$

Note que agora, faremos apenas $a \cdot e = A$ e desta forma, não estamos mudando a variável.

$$\int a^x \cdot e^x dx = \int (a \cdot e)^x dx = \int A^x dx = \frac{A^x}{\ln A} + c = \frac{(ae)^x}{\ln(ae)} + c$$

$$1) \int \frac{\cos x + \operatorname{sen} x}{\operatorname{sen}^3 x} dx$$

Resolução

$$\text{Veja } \int \frac{\cos x + \operatorname{sen} x}{\operatorname{sen}^3 x} dx = \int \left(\frac{\cos x}{\operatorname{sen}^3 x} + \frac{\operatorname{sen} x}{\operatorname{sen}^3 x} \right) dx = \int \left(\frac{\cos x}{\operatorname{sen}^3 x} + \frac{1}{\operatorname{sen}^2 x} \right) dx$$

$$\text{Note ainda que } \frac{1}{\operatorname{sen}^2 x} = \left(\frac{1}{\operatorname{sen} x} \right)^2 = (\operatorname{csc} x)^2 = \operatorname{csc}^2 x$$

$$\begin{aligned} \text{Logo: } \int \frac{\cos x + \operatorname{sen} x}{\operatorname{sen}^3 x} dx &= \int \left(\frac{\cos x}{\operatorname{sen}^3 x} + \frac{\operatorname{sen} x}{\operatorname{sen}^3 x} \right) dx = \int \left(\frac{\cos x}{\operatorname{sen}^3 x} + \frac{1}{\operatorname{sen}^2 x} \right) dx = \\ &= \int \left(\frac{\cos x}{\operatorname{sen}^3 x} + \operatorname{csc}^2 x \right) dx = \int \frac{\cos x}{(\operatorname{sen} x)^3} dx + \int \operatorname{csc}^2 x dx = \\ &= \int \frac{1}{(\operatorname{sen} x)^3} \cos x dx + \int \operatorname{csc}^2 x dx \end{aligned}$$

Vamos resolver a primeira integral: $\int \frac{1}{(\operatorname{sen} x)^3} \cos x dx$ e para isso devemos fazer

$$\operatorname{sen} x = u \Rightarrow (\operatorname{sen} x)' = \frac{du}{dx} \Rightarrow \cos x = \frac{du}{dx} \Rightarrow \cos x dx = du$$

$$\begin{aligned} \text{Donde } \int \frac{1}{(\operatorname{sen} x)^3} \cos x dx &= \int \frac{1}{u^3} du = \int u^{-3} du = \frac{u^{-2}}{-2} + c = -\frac{1}{2u^2} + c = \\ &= -\frac{1}{2(\operatorname{sen} x)^2} + c = -\frac{1}{2\operatorname{sen}^2 x} + c \end{aligned}$$

E agora a segunda integral que é $\int \operatorname{csc}^2 x dx = -\operatorname{ctg} x + c$

E finalmente as duas juntas:

$$\int \frac{1}{(\operatorname{sen} x)^3} \cos x dx + \int \operatorname{csc}^2 x dx = -\frac{1}{2\operatorname{sen}^2 x} - \operatorname{ctg} x + c$$

$$m) \int \frac{1}{1+4x^2} dx$$

Resolução

$$\text{Observe que } \int \frac{1}{1+4x^2} dx = \int \frac{1}{1+(2x)^2} dx$$

$$\text{Fazemos } 2x = u \Rightarrow (2x)' = \frac{du}{dx} \Rightarrow 2 = \frac{du}{dx} \Rightarrow 2dx = du \Rightarrow dx = \frac{1}{2} du$$

$$\begin{aligned} \text{Agora temos } \int \frac{1}{1+4x^2} dx &= \int \frac{1}{1+(2x)^2} dx = \int \frac{1}{1+u^2} \frac{1}{2} du = \frac{1}{2} \int \frac{1}{1+u^2} du = \\ &= \frac{1}{2} \operatorname{arc} \operatorname{tg} u + c = \frac{1}{2} \operatorname{arc} \operatorname{tg} (2x) + c \end{aligned}$$

$$n) \int \cos\left(\frac{3}{x}\right) \cdot \frac{1}{x^2} dx$$

Resolução

$$\int \cos\left(\frac{3}{x}\right) \cdot \frac{1}{x^2} dx = \int \cos(3x^{-1}) \cdot x^{-2} dx$$

$$\text{Agora fazemos } 3x^{-1} = u \Rightarrow (3x^{-1})' = \frac{du}{dx} \Rightarrow -3x^{-2} = \frac{du}{dx} \Rightarrow -3x^{-2} dx = du$$

$$-3x^{-2} dx = du \quad (-1) \Rightarrow 3x^{-2} dx = -du \Rightarrow x^{-2} dx = -\frac{1}{3} du$$

$$\text{E finalmente } \int \cos\left(\frac{3}{x}\right) \cdot \frac{1}{x^2} dx = \int \cos(3x^{-1}) \cdot x^{-2} dx = \int \cos u \cdot \left(-\frac{1}{3} du\right) =$$

$$= -\frac{1}{3} \int \cos u du = -\frac{1}{3} \operatorname{sen} u + c = -\frac{1}{3} \operatorname{sen}(3x^{-1}) + c = -\frac{1}{3} \operatorname{sen}\left(\frac{3}{x}\right) + c$$

$$o) \int \operatorname{tg}^2 x dx$$

Resolução

$$\text{Observe que } \operatorname{tg}^2 x = \frac{\operatorname{sen}^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 - 1 = (\sec x)^2 - 1$$

Ou seja $\operatorname{tg}^2 x = \sec^2 x - 1$, logo:

$$\int \operatorname{tg}^2 x dx = \int (\sec^2 x - 1) dx = \int \sec^2 x dx - \int dx = \operatorname{tg} x - x + c$$

$$p) \int (\operatorname{tg} \theta + \operatorname{ctg} \theta)^2 d\theta$$

Resolução

Observe que:

$$(\operatorname{tg} \theta + \operatorname{ctg} \theta)^2 = \operatorname{tg}^2 \theta + 2 \cdot \operatorname{tg} \theta \cdot \operatorname{ctg} \theta + \operatorname{ctg}^2 \theta = \operatorname{tg}^2 \theta + 2 \cdot \operatorname{tg} \theta \cdot \frac{1}{\operatorname{tg} \theta} + \operatorname{ctg}^2 \theta =$$

$$= \operatorname{tg}^2 \theta + 2 + \operatorname{ctg}^2 \theta = \underbrace{\operatorname{tg}^2 \theta + 1}_{\sec^2 \theta} + \underbrace{1 + \operatorname{ctg}^2 \theta}_{\csc^2 \theta} = \sec^2 \theta + \csc^2 \theta$$

$$\text{Logo, temos que } \int (\operatorname{tg} \theta + \operatorname{ctg} \theta)^2 d\theta = \int (\sec^2 \theta + \csc^2 \theta) d\theta = \int \sec^2 \theta d\theta + \int \csc^2 \theta d\theta =$$

$$\operatorname{tg} \theta + (-\operatorname{ctg} \theta) + c = \operatorname{tg} \theta - \operatorname{ctg} \theta + c$$

q) $\int \frac{1 + \operatorname{sen} x}{x - \cos x} dx$

Resolução

Fazemos $x - \cos x = u \Rightarrow (x - \cos x)' = \frac{du}{dx} \Rightarrow 1 - (-\operatorname{sen} x) = \frac{du}{dx} \Rightarrow$

$$1 + \operatorname{sen} x = \frac{du}{dx} \Rightarrow (1 + \operatorname{sen} x) dx = du$$

Agora, temos que

$$\int \frac{1 + \operatorname{sen} x}{x - \cos x} dx = \int \frac{1}{x - \cos x} (1 + \operatorname{sen} x) dx = \int \frac{1}{u} du = \ln |u| + c = \ln |x - \cos x| + c$$

r) $\int \frac{\operatorname{sen} x}{\sqrt{1 - \cos x}} dx$

Resolução

Fazemos $1 - \cos x = u \Rightarrow (1 - \cos x)' = \frac{du}{dx} \Rightarrow 0 - (-\operatorname{sen} x) = \frac{du}{dx} \Rightarrow$

$$\operatorname{sen} x = \frac{du}{dx} \Rightarrow \operatorname{sen} x dx = du$$

Agora temos que $\int \frac{\operatorname{sen} x}{\sqrt{1 - \cos x}} dx = \int \frac{1}{\sqrt{1 - \cos x}} \operatorname{sen} x dx = \int \frac{1}{\sqrt{u}} du =$

$$\int \frac{1}{u^{\frac{1}{2}}} du = \int u^{-\frac{1}{2}} du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{2}{1} \cdot u^{\frac{1}{2}} + c = 2 \cdot \sqrt{u} + c = 2\sqrt{1 - \cos x} + c$$

s) $\int \frac{e^t + 2}{e^t + 2t} dt$

Resolução

Observe que $\int \frac{e^t + 2}{e^t + 2t} dt = \int \frac{1}{e^t + 2t} \cdot (e^t + 2) dt$

Agora fazemos $e^t + 2t = u \Rightarrow (e^t + 2t)' = \frac{du}{dt} \Rightarrow e^t + 2 = \frac{du}{dt} \Rightarrow (e^t + 2) dt = du$

E finalmente $\int \frac{e^t + 2}{e^t + 2t} dt = \int \frac{1}{e^t + 2t} \cdot (e^t + 2) dt = \int \frac{1}{u} du = \ln |u| + c = \ln |e^t + 2t| + c$

$$t) \int \frac{1}{x \cdot \ln x} dx$$

Resolução

$$\text{Observe que podemos fazer } \int \frac{1}{x \cdot \ln x} dx = \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$$

$$\text{E fazemos ainda } \ln x = u \Rightarrow (\ln x)' = \frac{du}{dx} \Rightarrow \frac{1}{x} = \frac{du}{dx} \Rightarrow \frac{1}{x} dx = du$$

$$\int \frac{1}{x \cdot \ln x} dx = \int \frac{1}{\ln x} \cdot \frac{1}{x} dx = \int \frac{1}{u} du = \ln |u| + c = \ln |\ln x| + c$$

$$u) \int \frac{3}{(1 + \sqrt{x}) \cdot \sqrt{x}} dx$$

Resolução

$$\text{Note que } \int \frac{3}{(1 + \sqrt{x}) \cdot \sqrt{x}} dx = 3 \int \frac{1}{1 + \sqrt{x}} \cdot \frac{1}{\sqrt{x}} dx$$

$$\text{Fazemos } 1 + \sqrt{x} = u \Rightarrow \left(1 + x^{\frac{1}{2}}\right)' = \frac{du}{dx} \Rightarrow \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{du}{dx} \Rightarrow \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{du}{dx} \Rightarrow$$

$$\frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} dx = du \Rightarrow \frac{1}{2} \cdot \frac{1}{\sqrt{x}} dx = du \Rightarrow \frac{1}{\sqrt{x}} dx = 2du$$

$$\begin{aligned} \text{Logo } \int \frac{3}{(1 + \sqrt{x}) \cdot \sqrt{x}} dx &= 3 \int \frac{1}{1 + \sqrt{x}} \cdot \frac{1}{\sqrt{x}} dx = 3 \int \frac{1}{u} \cdot 2du = 3 \cdot 2 \int \frac{1}{u} du = 6 \int \frac{1}{u} du = \\ &= 6 \ln |u| + c = 6 \ln |1 + \sqrt{x}| + c \end{aligned}$$

$$v) \int \sqrt{3x^2 + 1} \cdot x dx$$

Resolução

$$\text{Fazendo } 3x^2 + 1 = u \Rightarrow (3x^2 + 1)' = \frac{du}{dx} \Rightarrow 6x = \frac{du}{dx} \Rightarrow 6x dx = du \Rightarrow x dx = \frac{1}{6} du$$

Agora

$$\int \sqrt{3x^2 + 1} \cdot x dx = \int \sqrt{u} \frac{1}{6} du = \frac{1}{6} \int u^{\frac{1}{2}} du = \frac{1}{6} \cdot \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{6} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{6} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + c =$$

$$= \frac{2}{18} \sqrt{u^3} + c = \frac{1}{9} \sqrt{u^3} + c = \frac{1}{9} \sqrt{(3x^2 + 1)^3} + c$$

$$w) \int \frac{2+3x}{\sqrt{1+4x+3x^2}} dx$$

Resolução

$$\text{Fazendo } 1+4x+3x^2 = u \Rightarrow (1+4x+3x^2)' = \frac{du}{dx} \Rightarrow 4+6x = \frac{du}{dx} \Rightarrow$$

$$2(2+3x) = \frac{du}{dx} \Rightarrow 2(2+3x) dx = du \Rightarrow (2+3x) dx = \frac{1}{2} du$$

$$\text{Agora temos que } \int \frac{2+3x}{\sqrt{1+4x+3x^2}} dx = \int \frac{1}{\sqrt{1+4x+3x^2}} (2+3x) dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} du =$$

$$\frac{1}{2} \int \frac{1}{u^{\frac{1}{2}}} du = \frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \cdot \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{1}{2} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{2} \cdot \frac{2}{1} \cdot u^{\frac{1}{2}} + c = u^{\frac{1}{2}} + c =$$

$$= \sqrt{u} + c = \sqrt{1+4x+3x^2} + c$$

$$x) \int \sec x dx$$

Resolução

Aqui precisamos fazer um pequeno artifício

$$\int \sec x dx = \int \sec x \cdot \left(\frac{\sec x + \operatorname{tg} x}{\sec x + \operatorname{tg} x} \right) dx = \int \frac{\sec^2 x + \sec x \cdot \operatorname{tg} x}{\sec x + \operatorname{tg} x} dx$$

$$\text{Agora fazemos } \sec x + \operatorname{tg} x = u \Rightarrow (\sec x + \operatorname{tg} x)' = \frac{du}{dx} \Rightarrow (\sec x)' + (\operatorname{tg} x)' = \frac{du}{dx}$$

$$\sec x \cdot \operatorname{tg} x + \sec^2 x = \frac{du}{dx} \Rightarrow (\sec x \cdot \operatorname{tg} x + \sec^2 x) dx = du$$

$$\text{E finalmente } \int \sec x dx = \int \sec x \cdot \left(\frac{\sec x + \operatorname{tg} x}{\sec x + \operatorname{tg} x} \right) dx = \int \frac{\sec^2 x + \sec x \cdot \operatorname{tg} x}{\sec x + \operatorname{tg} x} dx =$$

$$= \int \frac{1}{\sec x + \operatorname{tg} x} (\sec^2 x + \sec x \cdot \operatorname{tg} x) dx = \int \frac{1}{u} du = \ln |u| + c = \ln |\sec x + \operatorname{tg} x| + c$$